The Generalised Reflection Equation

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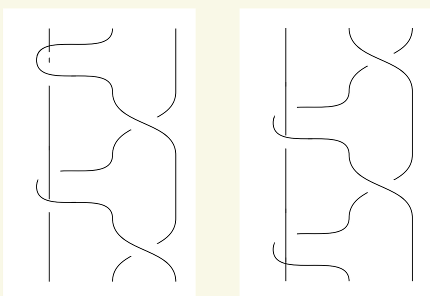
Aim

The aim of this project was to find all solutions to the generalised reflection equation for some finite dimensional Hopf algebras.

Introduction

Mathematics, especially the study of algebra, is focused on finding solutions to equations. Equations can come in many forms, and can be derived for a multitude of reasons. One such equation is named the reflection equation, whose name stems from one of its geometric interpretations. The derivation of the reflection equation comes from the following two braids:



The two braids are the same in the sense that it is possible to transform the left braid into the right braid without moving the ends of the strings. We capture this behaviour in the following reflection equation:

$$(K \otimes id)c(id \otimes K)c = c(K \otimes id)c(id \otimes K)$$

And its generalised form is expressed as:

$$\Delta(K) = (K \otimes 1)R_{2,1}(1 \otimes K)R$$

This poster will introduce the maths necessary to understand these equations, and showcase the solutions I found during the project.

Vector spaces

Vectors (as taught in an A-level maths or physics course) are typically depicted as arrows that have a magnitude and a direction. If we also have a set of axis defined, we usually represent a vector as a list of numbers. Mathematicians usually define a vector in a more general way that encapsulates many more objects under the umbrella term of vector.

A vector space is defined as a set along with two operations: scalar multiplication, and vector addition. Any member of this set will henceforth be called a vector. These two operations must satisfy the following properties:

- $(a + b) + c = a + (b + c)$ for any three vectors a, b, c.
- $a + b = b + a$ for any two vectors a, b.
- There is a special vector 0 that has the property $a + 0 = a$ for all vectors a.
- For every vector a, there is another vector $-a$ such that $a + (-a) = 0$.
- Given a vector a and the number 1, we have $1a = a$.
- Let λ be some number, and a, b be vectors, then $\lambda(a + b) = \lambda a + \lambda b$
- For the numbers λ, μ, and the vector a, $(\lambda\mu)a = \lambda(\mu a)$ and $(\lambda + \mu)a = \lambda a + \mu a$

Mathematically, we call a set of axes a "basis of the vector space". A basis is defined in the following way:
Let $v_1, v_2, \ldots, v_n$ be a set of n vectors in the vector space.

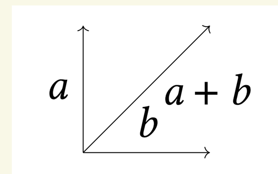
- We say that the set of vectors span the vector space if every vector can be put in the form $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$, for some selection of numbers $a_1, a_2, \ldots, a_n$.
- We say that the set of vectors is linearly independent if the only numbers $a_1, a_2, \ldots, a_n$ that satisfy the equation $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ are $a_1 = a_2 = \ldots = a_n = 0$.

A set of vectors is called a basis of the vector space if it spans the vector space, and is linearly independent.

Examples of vector spaces

1) The set of all possible lists of n numbers. This is normally what people think of when they hear about vectors. The vectors look like: $(a_1, a_2, \ldots, a_n)$ for numbers $a_1, a_2, \ldots, a_n$.

2) The set of all arrows originating from a point in 2 dimensional space. Scalar multiplication looks like scaling the length of the arrow, and addition is done like in this diagram:



3) The set of all quadratic polynomials of the form $ax^2 + bx + c$ with numbers a, b, c. This satisfies all of the conditions of a vector space.

Linear maps and tensor products

If we have two vector spaces V and W, we have a notion of a function, that takes in a vector in V, and gives out a vector in W. We call the function a linear map if the function f has the following properties:

- $f(a + b) = f(a) + f(b)$for any vectors a, b in V.
- $f(\lambda a) = \lambda f(a)$ for any vector a and number λ.

It can be seen from this definition that any spacial transformation (like a rotation or a reflection) can be described with a linear map. Linear maps can always be represented by a table of numbers called a matrix, and it can be proven that all matrices are linear maps.

Now we come to the tensor product. Given two vector spaces V, W, it is easy to see that we can construct a vector space where each vector has the form (v, w) where v is from V, and w is from W. This is called the Cartesian product of V and W, and is usually denoted by $V \times W$. Since the Cartesian product is a vector space, we should be able to construct a linear map out of it. However, if we assume the map is linear in both V and W, we get the following:

- $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ and $(v, w_1 + w_2) = (v, w_1) + (v, w_2)$
- $(\lambda v, w) = \lambda(v, w) = (v, \lambda w)$

If we use these equations to remove the redundant vectors, the space we are left with is called the tensor product of V and W denoted $V \otimes W$.
If we have vector spaces V and W, and we let $v_1, \ldots, v_n$ be a basis for V, and $w_1, \ldots, w_m$ be a basis for W, then $v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_n \otimes w_{m-1}, v_n \otimes w_m$ is a basis for $V \otimes W$

Algebras

If we have a vector space, we can define an operation μ that takes in two vectors, and gives out a third vector as a result. We will call this operation vector multiplication (or vector product) if the operation satisfies the following properties:

- $\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c))$ for any three vectors a, b, c.
- There is a vector η with $\mu(\eta \otimes a) = a = \mu(a \otimes \eta)$
- for the numbers α, β and the vectors a, b, we have $\mu((\alpha a) \otimes (\beta b)) = (\alpha\beta)\mu(a \otimes b)$
- for any three vectors a, b, c, we have $\mu((a + b) \otimes c) = \mu(a \otimes c) + \mu(b \otimes c)$ and $\mu(a \otimes (b + c)) = \mu(a \otimes b) + \mu(a \otimes c)$

A vector space with vector multiplication is called an algebra.

An example of an algebra is the algebra of polynomials of any degree, which as seen previously are vectors, but also have multiplication.

We often consider an algebra alongside the homomorphisms into the algebra of $n \times n$ matrices which is called a matrix representation of the algebra. This representation allows us to assign some spacial transformation to each vector in the algebra.

Hopf algebras and braided algebras

Similar to the vector product, we can also define a vector co-product. The vector co-product takes a vector in the vector space V, and returns a vector in the tensor product space $V \otimes V$. We call a vector space with a vector co-product a co-algebra.

Some algebras are both algebras and co-algebras at the same time, we call such an algebra a bi-algebra. Some bi-algebras also have a unique function called an antipode, which allows a bi-algebra the ability to describe symmetry. A bi-algebra with an antipode is called a Hopf algebra.

In all Hopf algebras, we can define the function $\Delta^{op}(x) = flip(\Delta(x))$ where $flip(x \otimes y) = y \otimes x$. There are some Hopf algebras where $\Delta(x) = \Delta^{op}(x)$, and there are some where there is no relation. But sometimes we have that $\Delta^{op}(x) = R^{-1}\Delta(x)R$ for some vector R. If we add a few more conditions to R, we call R a universal R-matrix. An algebra with a universal R-matrix is called a Braided algebra. We also define the c from the reflection equation by $c = flip(R)$.

Any vector K in a braided algebra that solves the reflection equation is called a K-matrix, and any vector K that solves the generalised reflection equation is called a universal K-matrix.

Results

Sweedler's Hopf-algebra

A vector in this algebra looks like $a + bx + cy + dxy$, and uses the following properties:

- $xy = -yx$
- $y^2 = 0$
- $x^2 = 1$

Theorem: *The only 2 universal K-matrices are x and 1.*

The group algebra $\mathbb{C}[G]$

A group is a set of objects, along with a notion of multiplication satisfying the following conditions:

- $(a * b) * c = a * (b * c)$
- $e * a = a * e = a$
- $a * a^{-1} = e$

A vector in this algebra is of the form $\sum_g a_g g$ where g is a member of the group.

Theorem: *The universal K-matrices of $\mathbb{C}[G]$ are of the form $K = g$ where g is a group element.*

The quantum double algebra $D(\mathbb{C}[G])$

A vector in this algebra looks like $\sum_{g,h} a_{g,h} g \otimes \chi_h$ where g and h are in the group, a is a number, and χ_g is a function which gives the output of 1 when its input is g, and gives an output of 0 otherwise.

Theorem: *The universal K-matrices are of the form $K = \sum_x \rho(x)x \otimes \chi_{kx}$ for any element k with the property of $kg = gk$ for all elements in the group, and $\rho(x)$ being a one dimensional representation of $\mathbb{C}[G]$.*

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